# DIRECT PROOFS OF LEGENDRE AND ANDRICA CONJECTURES

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**Abstract**: In this paper direct proofs of Andrica and Legendre conjectures are presented using a direct proof method. The proof method will use the implication representing gaps between consecutive odd primes in terms of number of composite even numbers in conjunction with the prime number theorem.

## 1. Introduction

The Legendre conjecture was proposed by Adrien- Marie Legendre (1752-1833) and states that there is a prime number between two successive square natural numbers [1]. The conjecture is one of the Landau's problems. The conjecture states that given an integer $n > 0$, there exists a prime number between $n^2$ and $(n+1)^2$.

Andrica conjecture [2] states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, where $p_n$ is the nth prime umber, holds for all $n \geq 1$

## 2. Notation

In this paper $p_n$ represents the n$^{th}$ prime number; $g_n$ represents the gap between $p_n$ and $p_{n+1}$ numbers; $N_{co}$ represents the number of composite odd numbers between $p_n$ and $p_{n+1}$.

## 3. Premises

### 3.1 Premise on gap between successive odd primes

The gap between successive primes is given by (1) below:

$$g_n = 2N_{co} + 2 \qquad (1)$$

### 3.2 Premise on $\frac{(\ln x)^2}{x}$ ratio

For a positive integer, x, the largest $\frac{(\ln x)^2}{x}$ is $\frac{(\ln 7)^2}{7}$

## 4. Direct proof of Andrica conjecture

Andrica conjecture stipulates that:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \qquad (2)$$

The equation (1) has the implication that inequality (2) can be reformulated as in (3) below:

$$\sqrt{p_n + 2N_{co} + 2} - \sqrt{p_n} < \frac{N_{co}+1}{\sqrt{p_n}} = \frac{g_n}{2\sqrt{p_n}} \qquad (3)$$

From inequality (3) the next big task is to determine the maximum value of $\frac{g_n}{\sqrt{p_n}}$. The prime number theory should provide a way through the inequality (4) below that holds for any positive integer x.

$$\frac{(\ln x)^2}{x} \leq \frac{(\ln 7)^2}{7} \qquad (4)$$

The inequality (4) has implication (5) on (3):

$$\sqrt{p_n + 2N_{co} + 2} - \sqrt{p_n} < \frac{N_{co}+1}{\sqrt{p_n}} = \frac{g_n}{2\sqrt{p_n}} \leq \frac{2}{\sqrt{7}} \qquad (5)$$

The inequality (5) means that:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{2}{\sqrt{7}} \qquad (6)$$

This proves the Andrica conjecture. The inequality (6) can be rewritten as in (7) below:

$$g_n < \frac{4}{7} + \frac{4}{\sqrt{7}}\sqrt{p_n} \qquad (7)$$

The gap of inequality (7) can be further reduced by substituting $p_n = 2$ in the expression $\frac{N_{co}+1}{\sqrt{2}}$, in which case it takes the form 8 below:

$$g_n < \tfrac{1}{2} + \sqrt{2p_n} \qquad (8)$$

## 5. Direct proof of Legendre conjecture

If n is a positive integer, the Legendre conjecture stipulates that there exists a prime between $n^2$ and $(n+1)^2$ .

Now:

$$\sqrt{(n+1)^2 - n^2} = 1 \qquad (9)$$

Now if:

$$p_n = n^2 + r \qquad (10)$$

$$p_{n+1} = (n+1)^2 - s \qquad (11)$$

By the above proved Andrica conjecture:

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{(n+1)^2 - s} - \sqrt{n^2 + r} < \frac{2}{\sqrt{7}} \qquad (12)$$

The inequality (12) implies that there are at least two primes in the gap between $n^2$ and $(n+1)^2$ . This proves the Legendre conjecture.

## 6. Conclusion

Andrica and Legendre conjectures have been proved by direct proof method. The Andrica conjecture implies Legendre conjecture.

## References


[1] Jing-Run, C. (1975). Of division of almost Primes in an Interval. *Scientia Sinica, Vol. 18(5)* , 611-627.

[2] Andrica, D. (1986). Note on Conjecture in Prime Number Theory. *Studia Univ. Babes-Bolai, Math. (31)* , 44-48.